$$\iint_D e^{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^1 e^{r^2}\,r\,dr\,d\theta \overset{u=r^2}{=} \int_0^{2\pi}\int_0^1 \frac{1}{2}e^u\,du\,d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}(e-1)\,d\theta = \pi(e-1).$$

$\Diamond$

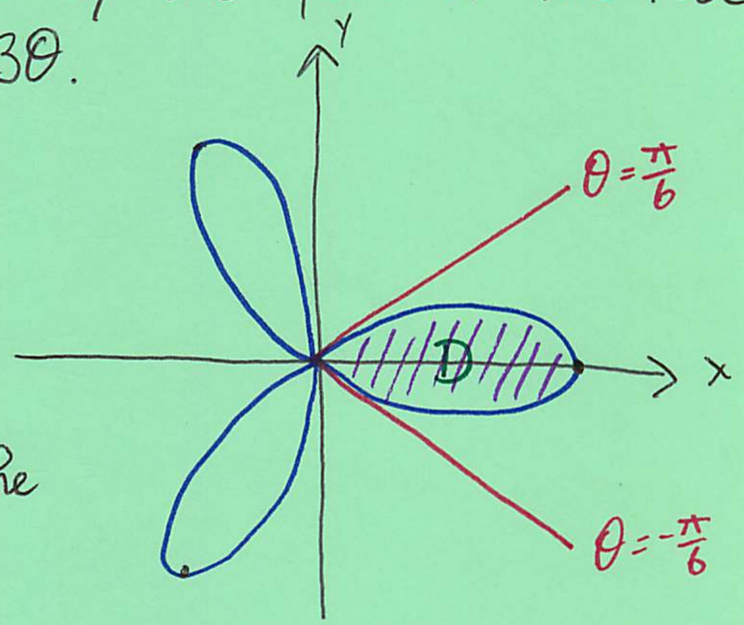
Lecture 14

Regions of integration need not be polar rectangles, naturally. Consider the following problem from Calc II:

Ex: Find the area enclosed by one petal of the rose

$$r = \cos 3\theta.$$

Sol: The rose looks like:



We know $\cos 3\theta = 0$ when $3\theta = \frac{\pi}{2} + n\pi \iff \theta = \frac{\pi}{6} + \frac{n\pi}{3}$.

Taking $-\frac{\pi}{6} \le \theta \le \frac{\pi}{6}$, we get the indicated petal. So, to get the bounds on r, we fix a θ-value (a ray coming out of the origin) and find an "inner" and "outer" bound on r: in this case, $0 \le r \le \cos 3\theta$. So,

$$A(D) = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}}\int_0^{\cos 3\theta} r\,dr\,d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2}\cos^2 3\theta\,d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{4}(1+\cos 6\theta)\,d\theta$$

$$= \frac{1}{4}\left(\theta + \frac{1}{6}\sin 6\theta\right)\Big|_{-\frac{\pi}{6}}^{\frac{\pi}{6}} = \frac{1}{4}\left[\left(\frac{\pi}{6}+\frac{1}{6}\sin\pi\right) - \left(\frac{-\pi}{6}+\frac{1}{6}\sin(-\pi)\right)\right]$$

$$= \frac{\pi}{12}$$

$\Diamond$

Ex: Set up an integral giving the volume of the region bounded above by the paraboloid $z=4-x^2-y^2$, below by the xy-plane, and inside the cylinder $x^2+y^2=2y$.

Sol: First, we rewrite these in polar coordinates:

paraboloid: $z=4-r^2$, cylinder: $r^2=2r\sin\theta$

$\Leftrightarrow r=2\sin\theta$

In the cylinder, $r=0 \Leftrightarrow 2\sin\theta=0 \Leftrightarrow \theta=\pi+n\pi$

So, letting θ go from 0 to π, we get the cylinder.

The bounds on r go from 0 to $2\sin\theta$. Thus:

$$\text{Vol}=\iint_D z\,dA=\int_0^{\pi}\int_0^{2\sin\theta}(4-r^2)\,r\,dr\,d\theta=\int_0^{\pi}\int_0^{2\sin\theta}(4r-r^3)\,dr\,d\theta$$

$$=\int_0^{\pi}(8\sin^2\theta-4\sin^4\theta)\,d\theta=\int_0^{\pi}\left(\frac{5}{2}-2\cos 2\theta-\frac{1}{2}\cos 4\theta\right)d\theta$$

$$=10\pi$$

$\diamond$

15.7 - Triple Integrals

As you might imagine, triple integrals are defined using a triple Riemann sum. We'll leave the details of that to the book.

Let's start with the most basic example:

Ex: Compute the triple integral of $f(x,y,z) = x^2 y e^{xyz}$ over the box $B = [0,1] \times [1,2] \times [2,3]$.

Sol: $\iiint_B x^2 y e^{xyz}\, dV = \int_0^1 \int_1^2 \int_2^3 x^2 y e^{xyz}\, dz\, dy\, dx$

$= \int_0^1 \int_1^2 x e^{xyz} \Big|_2^3 dy\, dx = \int_0^1 \int_1^2 (xe^{3xy} - xe^{2xy})\, dy\, dx$

$= \int_0^1 \left(\frac{1}{3}e^{3xy} - \frac{1}{2}e^{2xy}\right)\Big|_1^2 dx = \int_0^1 \left[\left(\frac{1}{3}e^{6x} - \frac{1}{2}e^{4x}\right) - \left(\frac{1}{3}e^{3x} - \frac{1}{2}e^{2x}\right)\right] dx$

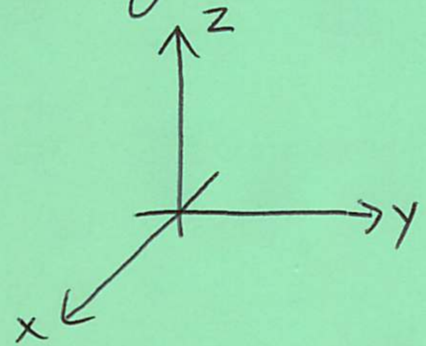
$= \int_0^1 \left(\frac{1}{3}e^{6x} - \frac{1}{2}e^{4x} - \frac{1}{3}e^{3x} + \frac{1}{2}e^{2x}\right) dx$

$= \left(\frac{1}{18}e^{6x} - \frac{1}{8}e^{4x} - \frac{1}{9}e^{3x} + \frac{1}{4}e^{2x}\right)\Big|_0^1$

$= \frac{e^6}{18} - \frac{e^4}{8} - \frac{e^3}{9} + \frac{e^2}{4} - \left(\frac{1}{18} - \frac{1}{8} - \frac{1}{9} + \frac{1}{4}\right)$

$\diamond$

There is, of course, no reason to stick to boxes. Let's say our region is E. Let's orient the axes as such:



Then, when setting up bounds, they look as follows

$\underline{x}$: "back to front"

$\underline{y}$: "left to right"

$\underline{z}$: "bottom to top"

Again, sketching the region will be important!

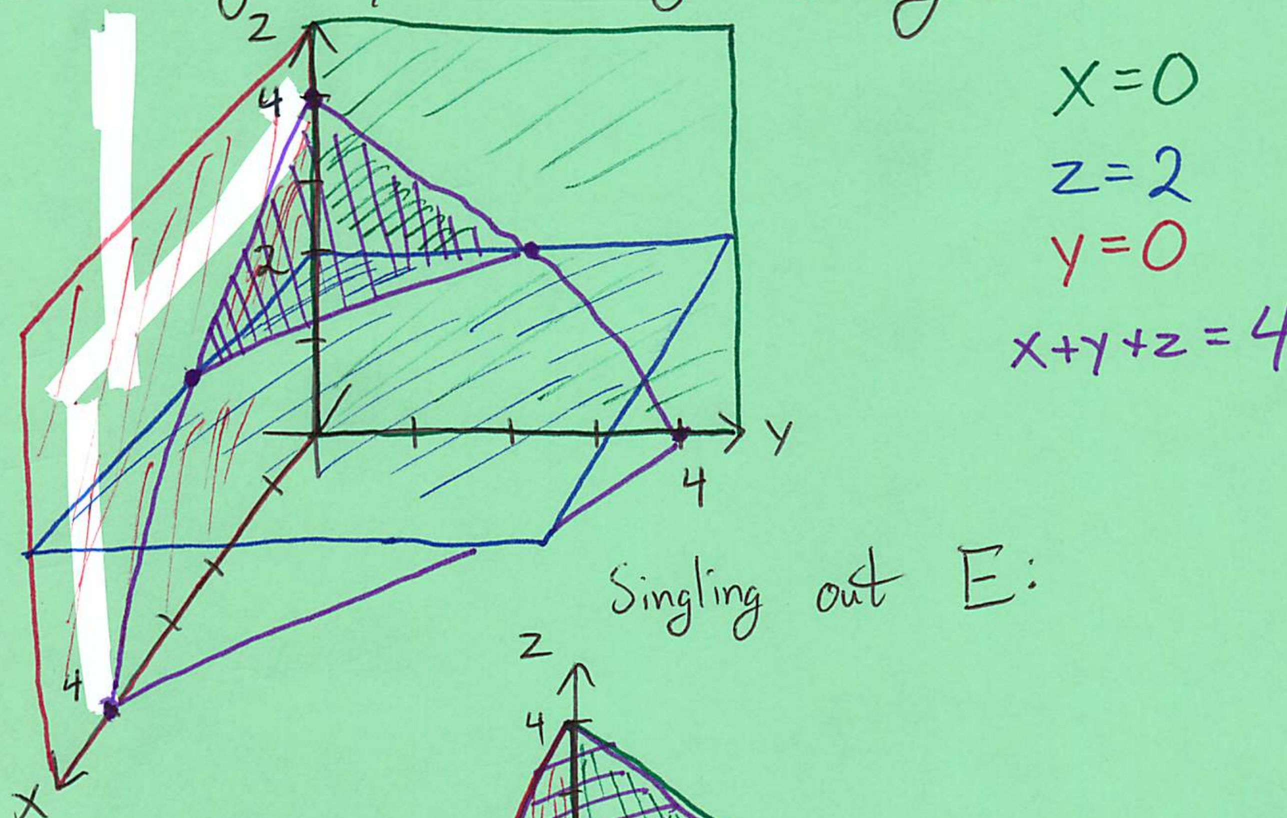
Now, once we've figured out the bounds on the inside integral, the outer two integrals' bounds come from setting up a double integral over a "shadow" region:

If the inside integral is respect to x / y / z, then we look at the shadow of E in the yz / xz / xy -plane and set up the double integral over that.

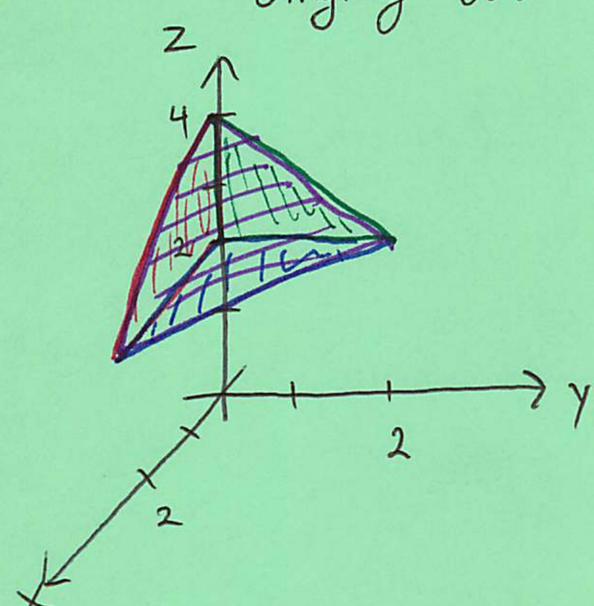
Let's make this concrete with an example.

Ex: Set up the integral to compute the volume of E, where E is the tetrahedron bounded by the planes: $x=0$, $y=0$, $z=2$, and $x+y+z=4$.

Sol: Begin by sketching the region:

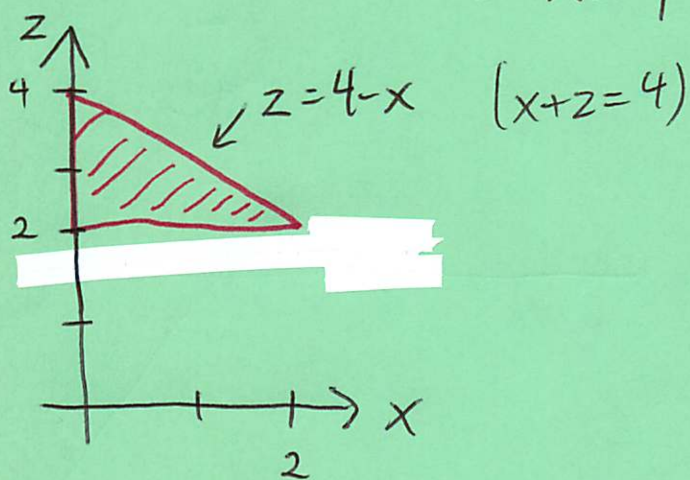


Singling out E:



Now, we need an order of integration. Let's integrate y first... because, why not?!

So, we look from left to right and see that the left function is $y=0$ and the right function is $x+y+z=4 \iff y=4-x-z$. So the inside integral is: $\int_0^{4-x-z} dV$. The shadow E makes in the xz-plane is:



So, the volume is:

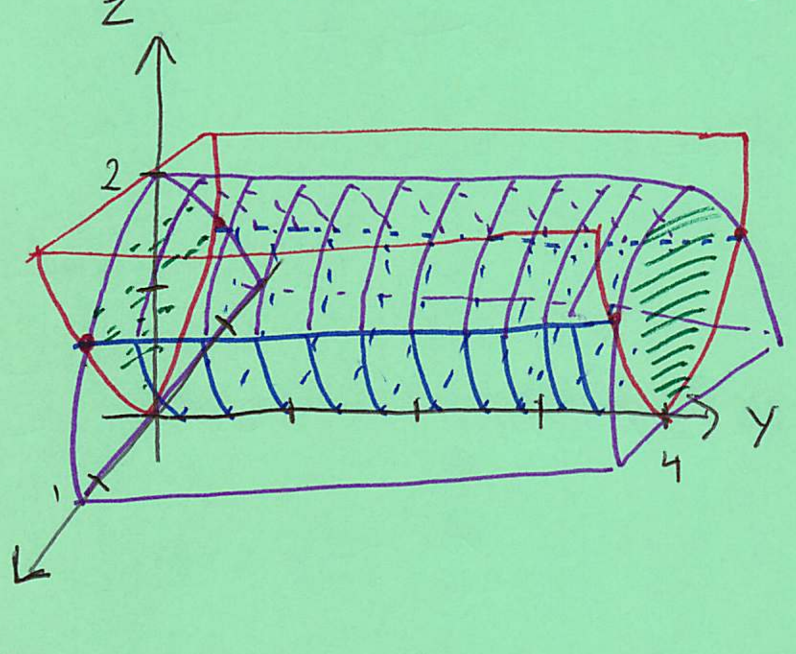
$$\mathrm{Vol}(E) = \iiint_E dV = \int_0^2 \int_0^{4-x} \int_0^{4-x-z} dy\,dz\,dx$$

$\diamond$

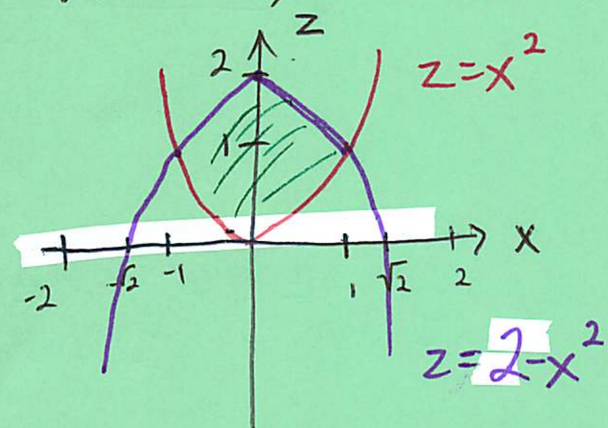
As with double integrals, we may need to switch the order of integration.

<u>Ex</u>: Rewrite $\int_0^4 \int_{-1}^1 \int_{x^2}^{2-x^2} xyz\,dz\,dx\,dy$ using $dy\,dz\,dx$.

Sol: We begin by sketching the region.



It's simple to see here that y goes from 0 to 4. Now, the shadow in the xz-plane is:



So, $\int_0^4\int_{-1}^1\int_{x^2}^{2-x^2} xyz\,dz\,dx\,dy = \int_{-1}^1\int_{2-x^2}^{x^2}\int_0^4 xyz\,dy\,dz\,dx$ ◊

Let's detour back to 15.5 for an application:

15.5 - Mass and Center of Mass

Recall that the center of mass of a system is the point where, if all of the mass of the system were concentrated there, the total moment is the same as in the original.

In the discrete case of masses m_i at points (x_i, y_i), the ~~total~~ moment, about the

- y-axis is $M_y = \sum_i m_i x_i$ → moment of mass
- x-axis is $M_x = \sum_i m_i y_i$

So, the center of mass has coordinates $(\bar{x}, \bar{y})$ where $(\sum_i m_i)\bar{x} = \sum_i m_i x_i$ & $(\sum_i m_i)\bar{y} = \sum_i m_i y_i$.

Thus, $\bar{x} = \dfrac{M_y}{\text{total mass}}$, $\bar{y} = \dfrac{M_x}{\text{total mass}}$

We can do this for a system in $\mathbb{R}^3$ as well, e.g. $\bar{z} = \dfrac{M_{xy}}{\text{total mass}}$, where

$M_{xy} = \sum_i m_i z_i$ is the total moment about the xy-plane.

Continuous Case: Let's suppose we had a lamina D in $\mathbb{R}^2$ with density function $\rho(x,y)$. Then, the total mass of D is

$$\boxed{\text{mass} = m = \iint_D \rho(x,y)\,dA.}$$

Using the discrete case as a guide, we find that

$$M_y = \iint_D x\rho(x,y)\,dA \quad \& \quad M_x = \iint_D y\rho(x,y)\,dA.$$

So, the center of mass of D has coordinates:

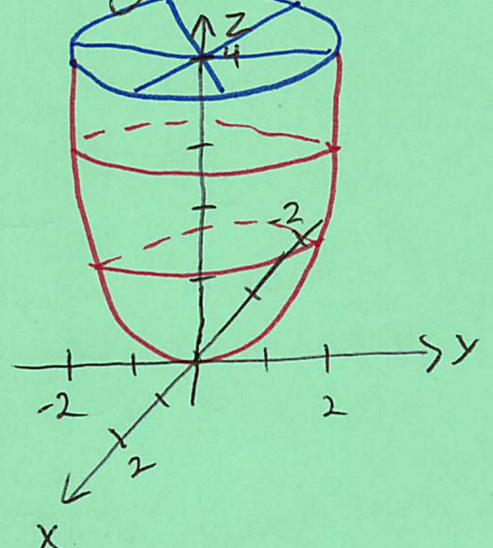
$$\boxed{(\bar{x}, \bar{y}) = \left(\frac{\iint_D x\rho\,dA}{\iint_D \rho\,dA}, \frac{\iint_D y\rho\,dA}{\iint_D \rho\,dA} \right)}.$$

Similar equations hold for solid regions in $\mathbb{R}^3$.

15.8 - Cylindrical Coordinates

Ex: Compute the volume of the solid bounded by $z = x^2 + y^2$ and $z = 4$.

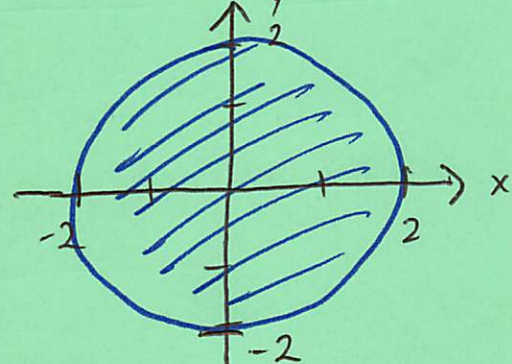
Sol: Sketch the region:



It looks easiest to start by integrating z first:

$$\text{Vol} = \int_?^? \int_?^? \int_{x^2+y^2}^{4} dz\, d?\, d?$$

The shadow of this region in the xy-plane is just the disk of radius 2:



We could set up bounds in cartesian here, but since this is a disk, polar coordinates are easiest. The function we're integrating over the disk is

$$f(x,y) = \int_{x^2+y^2}^{4} dz \iff f(r\cos\theta, r\sin\theta) = \int_{r^2}^{4} dz$$

So, we should have:

$$\text{Vol} = \int_0^{2\pi}\int_0^2 \left(\int_{r^2}^4 dz\right) r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 z\Big|_{r^2}^{4} r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 (4r - r^3)\,dr\,d\theta = \int_0^{2\pi} \left(2r^2 - \tfrac{1}{4}r^4\right)\Big|_0^2 d\theta$$

$$= \int_0^{2\pi} (8-4)\,d\theta = 8\pi.$$

◇